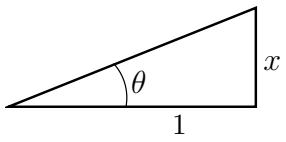
Basic Stuff

1.1 Trigonometry

The common trigonometric functions are familiar to you, but do you know some of the tricks to remember (or to derive quickly) the common identities among them? Given the sine of an angle, what is its tangent? Given its tangent, what is its cosine? All of these simple but occasionally useful relations can be derived in about two seconds if you understand the idea behind one picture. Suppose for example that you know the tangent of θ, what is $\sin\theta$? Draw a right triangle and designate the tangent of θ as x, so you can draw a triangle with $\tan\theta = x/1$.

The Pythagorean theorem says that the third side is $\sqrt{1+x^2}$. You now read the sine from the triangle as $x/\sqrt{1+x^2}$, so

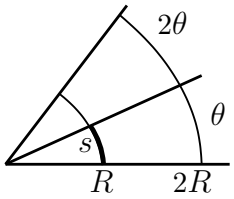


$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

Any other such relation is done the same way. You know the cosine, so what's the cotangent? Draw a different triangle where the cosine is $x/1$.

Radians

When you take the sine or cosine of an angle, what units do you use? Degrees? Radians? Cycles? And who invented radians? Why is this the unit you see so often in calculus texts? That there are 360° in a circle is something that you can blame on the Sumerians, but where did this other unit come from?



It results from one figure and the relation between the radius of the circle, the angle drawn, and the length of the arc shown. If you remember the equation $s = R\theta$, does that mean that for a full circle $\theta = 360^\circ$ so $s = 360R$? No. For some reason this equation is valid only in radians. The reasoning comes down to a couple of observations. You can see from the drawing that s is proportional to θ — double θ and you double s. The same observation holds about the relation between s and R, a direct proportionality. Put these together in a single equation and you can conclude that

$$s = CR\,\theta$$

where C is some constant of proportionality. Now what is C?

You know that the whole circumference of the circle is $2\pi R$, so if $\theta = 360^\circ$, then

$$2\pi R = CR\,360^\circ, \qquad \text{and} \qquad C = \frac{\pi}{180}\,\text{degree}^{-1}$$

It has to have these units so that the left side, s, comes out as a length when the degree units cancel. This is an awkward equation to work with, and it becomes *very* awkward when you try to do calculus. An increment of one in $\Delta\theta$ is big if you're in radians, and small if you're in degrees, so it should be no surprise that $\Delta\sin\theta/\Delta\theta$ is much smaller in the latter units:

$$\frac{d}{d\theta}\sin\theta = \frac{\pi}{180}\cos\theta \qquad \text{in degrees}$$

This is the reason that the radian was invented. The radian is the unit designed so that the proportionality constant is one.

$$C = 1\,\text{radian}^{-1} \qquad \text{then} \qquad s = \big(1\,\text{radian}^{-1}\big)R\theta$$

In practice, no one ever writes it this way. It's the custom simply to omit the C and to say that $s = R\theta$ with θ restricted to radians — it saves a lot of writing. How big is a radian? A full circle has circumference $2\pi R$, and this equals $R\theta$ when you've taken C to be one. It says that the angle for a full circle has 2π radians. One radian is then $360/2\pi$ degrees, a bit under $60°$. Why do you always use radians in calculus? Only in this unit do you get simple relations for derivatives and integrals of the trigonometric functions.

Hyperbolic Functions

The circular trigonometric functions, the sines, cosines, tangents, and their reciprocals are familiar, but their hyperbolic counterparts are probably less so. They are related to the exponential function as

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}, \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{1.1}$$

The other three functions are

$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \operatorname{csch} x = \frac{1}{\sinh x}, \qquad \coth x = \frac{1}{\tanh x}$$

Drawing these is left to problem 1.4, with a stopover in section 1.8 of this chapter.

Just as with the circular functions there are a bunch of identities relating these functions. For the analog of $\cos^2\theta + \sin^2\theta = 1$ you have

$$\cosh^2\theta - \sinh^2\theta = 1 \tag{1.2}$$

For a proof, simply substitute the definitions of $\cosh$ and $\sinh$ in terms of exponentials and watch the terms cancel. (See problem 4.23 for a different approach to these functions.) Similarly the other common trig identities have their counterpart here.

$$1 + \tan^2\theta = \sec^2\theta \qquad \text{has the analog} \qquad 1 - \tanh^2\theta = \operatorname{sech}^2\theta \tag{1.3}$$

The reason for this close parallel lies in the complex plane, because $\cos(ix) = \cosh x$ and $\sin(ix) = i\sinh x$. See chapter three.

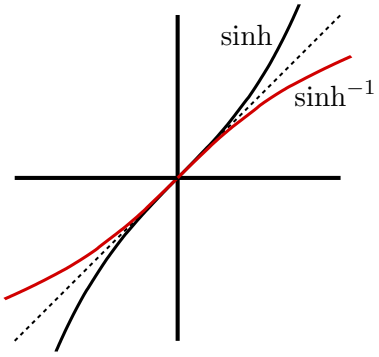
The inverse hyperbolic functions are easier to evaluate than are the corresponding circular functions. I'll solve for the inverse hyperbolic sine as an example

$$y = \sinh x \qquad \text{means} \qquad x = \sinh^{-1} y, \qquad\qquad y = \frac{e^x - e^{-x}}{2}, \quad \text{solve for } x.$$

Multiply by $2e^x$ to get the quadratic equation

$$2e^x y = e^{2x} - 1 \qquad \text{or} \qquad \big(e^x\big)^2 - 2y\big(e^x\big) - 1 = 0$$

The solutions to this are $e^x = y \pm \sqrt{y^2+1}$, and because $\sqrt{y^2+1}$ is always greater than $|y|$, you must take the positive sign to get a positive e^x. Take the logarithm of e^x and



$$x = \sinh^{-1} y = \ln\left(y + \sqrt{y^2+1}\right)$$
$$(-\infty < y < +\infty)$$

As x goes through the values $-\infty$ to $+\infty$, the values that $\sinh x$ takes on go over the range $-\infty$ to $+\infty$. This implies that the domain of $\sinh^{-1} y$ is $-\infty < y < +\infty$. The graph of an inverse function is the mirror image of the original function in the $45°$ line $y = x$, so if you have sketched the graphs of the original functions, the corresponding inverse functions are just the reflections in this diagonal line.

The other inverse functions are found similarly; see problem 1.3

$$\begin{aligned}
\sinh^{-1} y &= \ln\left(y + \sqrt{y^2+1}\right) \\
\cosh^{-1} y &= \ln\left(y \pm \sqrt{y^2-1}\right), \qquad y \geq 1 \\
\tanh^{-1} y &= \frac{1}{2}\ln\frac{1+y}{1-y}, \qquad |y| < 1 \\
\coth^{-1} y &= \frac{1}{2}\ln\frac{y+1}{y-1}, \qquad |y| > 1
\end{aligned} \tag{1.4}$$

The $\cosh^{-1}$ function is commonly written with only the $+$ sign before the square root. What does the other sign do? Draw a graph and find out. Also, what happens if you add the two versions of the $\cosh^{-1}$?

The calculus of these functions parallels that of the circular functions.

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

Similarly the derivative of $\cosh x$ is $\sinh x$. Note the plus sign here, not minus.

Where do hyperbolic functions occur? If you have a mass in equilibrium, the total force on it is zero. If it's in *stable* equilibrium then if you push it a little to one side and release it, the force will push it back to the center. If it is *unstable* then when it's a bit to one side it will be pushed farther away from the equilibrium point. In the first case, it will oscillate about the equilibrium position and for small oscillations the function of time will be a circular trigonometric function — the common sines or cosines of time, $A\cos\omega t$. If the point is unstable, the motion will be described by hyperbolic functions of time, $\sinh\omega t$ instead of $\sin\omega t$. An ordinary ruler held at one end will swing back and forth, but if you try to balance it at the other end it will fall over. That's the difference between $\cos$ and $\cosh$. For a deeper understanding of the relation between the circular and the hyperbolic functions, see section 3.3

1.2 Parametric Differentiation

The integration techniques that appear in introductory calculus courses include a variety of methods of varying usefulness. There's one however that is for some reason not commonly done in calculus courses: *parametric differentiation.* It's best introduced by an example.

$$\int_0^\infty x^n e^{-x}\,dx$$

You could integrate by parts n times and that will work. For example, $n = 2$:

$$= -x^2e^{-x}\Big|_0^\infty + \int_0^\infty 2xe^{-x}\,dx = 0 - 2xe^{-x}\Big|_0^\infty + \int_0^\infty 2e^{-x}\,dx = 0 - 2e^{-x}\Big|_0^\infty = 2$$

Instead of this method, do something completely different. Consider the integral

$$\int_0^\infty e^{-\alpha x}\,dx \tag{1.5}$$

It has the parameter α in it. The reason for this will be clear in a few lines. It is easy to evaluate, and is

$$\int_0^\infty e^{-\alpha x}\,dx = \frac{1}{-\alpha}e^{-\alpha x}\Big|_0^\infty = \frac{1}{\alpha}$$

Now differentiate this integral with respect to α,

$$\frac{d}{d\alpha}\int_0^\infty e^{-\alpha x}\,dx = \frac{d}{d\alpha}\frac{1}{\alpha} \qquad \text{or} \qquad -\int_0^\infty xe^{-\alpha x}\,dx = \frac{-1}{\alpha^2}$$

And again and again: $\displaystyle +\int_0^\infty x^2e^{-\alpha x}\,dx = \frac{+2}{\alpha^3}, \qquad -\int_0^\infty x^3e^{-\alpha x}\,dx = \frac{-2\cdot 3}{\alpha^4}$

The n^{th} derivative is

$$\pm\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{\pm n!}{\alpha^{n+1}} \tag{1.6}$$

Set $\alpha = 1$ and you see that the original integral is $n!$. This result is compatible with the standard definition for $0!$. From the equation $n! = n\cdot(n-1)!$, you take the case $n = 1$, and it requires $0! = 1$ in order to make any sense. This integral gives the same answer for $n = 0$.

The idea of this method is to change the original problem into another by introducing a parameter. Then differentiate with respect to that parameter in order to recover the problem that you really want to solve. With a little practice you'll find this easier than partial integration. Also see problem 1.47 for a variation on this theme.

Notice that I did this using definite integrals. If you try to use it for an integral without limits you can sometimes get into trouble. See for example problem 1.42.

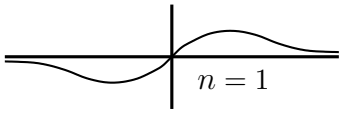
1.3 Gaussian Integrals

Gaussian integrals are an important class of integrals that show up in kinetic theory, statistical mechanics, quantum mechanics, and any other place with a remotely statistical aspect.

$$\int dx\, x^n e^{-\alpha x^2}$$

The simplest and most common case is the definite integral from $-\infty$ to $+\infty$ or maybe from 0 to ∞.

If n is a positive odd integer, these are elementary,



$$\int_{-\infty}^{\infty} dx\, x^n e^{-\alpha x^2} = 0 \qquad (n \text{ odd}) \tag{1.7}$$

To see why this is true, sketch graphs of the integrand for a few more odd n.

For the integral over positive x and still for odd n, do the substitution $t = \alpha x^2$.

$$\int_0^{\infty} dx\, x^n e^{-\alpha x^2} = \frac{1}{2\alpha^{(n+1)/2}} \int_0^{\infty} dt\, t^{(n-1)/2} e^{-t} = \frac{1}{2\alpha^{(n+1)/2}} \big((n-1)/2\big)! \tag{1.8}$$

Because n is odd, $(n-1)/2$ is an integer and its factorial makes sense.

If n is even then doing this integral requires a special preliminary trick. Evaluate the special case $n = 0$ and $\alpha = 1$. Denote the integral by I, then

$$I = \int_{-\infty}^{\infty} dx\, e^{-x^2}, \qquad \text{and} \qquad I^2 = \left(\int_{-\infty}^{\infty} dx\, e^{-x^2}\right)\left(\int_{-\infty}^{\infty} dy\, e^{-y^2}\right)$$

In squaring the integral you must use a different label for the integration variable in the second factor or it will get confused with the variable in the first factor. Rearrange this and you have a conventional double integral.

$$I^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-(x^2+y^2)}$$

This is something that you can recognize as an integral over the entire x-y plane. Now the trick is to switch to polar coordinates*. The element of area $dx\, dy$ now becomes $r\, dr\, d\phi$, and the respective limits on these coordinates are 0 to ∞ and 0 to 2π. The exponent is just $r^2 = x^2 + y^2$.

$$I^2 = \int_0^{\infty} r\, dr \int_0^{2\pi} d\phi\, e^{-r^2}$$

The ϕ integral simply gives 2π. For the r integral substitute $r^2 = z$ and the result is $1/2$. [Or use Eq. (1.8).] The two integrals together give you π.

$$I^2 = \pi, \qquad \text{so} \qquad \int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi} \tag{1.9}$$

Now do the rest of these integrals by parametric differentiation, introducing a parameter with which to carry out the derivatives. Change e^{-x^2} to $e^{-\alpha x^2}$, then in the resulting integral change variables to reduce it to Eq. (1.9). You get

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}, \qquad \text{so} \qquad \int_{-\infty}^{\infty} dx\, x^2 e^{-\alpha x^2} = -\frac{d}{d\alpha}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2}\left(\frac{\sqrt{\pi}}{\alpha^{3/2}}\right) \tag{1.10}$$

You can now get the results for all the higher even powers of x by further differentiation with respect to α.

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* See section 1.7 in this chapter

1.4 erf and Gamma

What about the same integral, but with other limits? The odd-n case is easy to do in just the same way as when the limits are zero and infinity: just do the same substitution that led to Eq. (1.8). The even-n case is different because it can't be done in terms of elementary functions. It is used to define an entirely new function.

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{-t^2} \tag{1.11}$$

x	0.	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
erf	0.	0.276	0.520	0.711	0.843	0.923	0.967	0.987	0.995

This is called the error function. It's well studied and tabulated and even shows up as a button on some* pocket calculators, right along with the sine and cosine. (Is erf odd or even or neither?) (What is $\text{erf}(\pm\infty)$?)

A related integral worthy of its own name is the Gamma function.

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t} \tag{1.12}$$

The special case in which x is a positive integer is the one that I did as an example of parametric differentiation to get Eq. (1.6). It is

$$\Gamma(n) = (n-1)!$$

The factorial is not defined if its argument isn't an integer, but the Gamma function is perfectly well defined for any argument as long as the integral converges. One special case is notable: $x = 1/2$.

$$\Gamma(1/2) = \int_0^\infty dt\, t^{-1/2} e^{-t} = \int_0^\infty 2u\, du\, u^{-1} e^{-u^2} = 2\int_0^\infty du\, e^{-u^2} = \sqrt{\pi} \tag{1.13}$$

I used $t = u^2$ and then the result for the Gaussian integral, Eq. (1.9). You can use parametric differentiation to derive a simple and useful recursion relation. (See problem 1.14 or 1.47.)

$$x\Gamma(x) = \Gamma(x+1) \tag{1.14}$$

From this you can get the value of $\Gamma(1\frac{1}{2})$, $\Gamma(2\frac{1}{2})$, etc. In fact, if you know the value of the function in the interval between one and two, you can use this relationship to get it anywhere else on the axis. You already know that $\Gamma(1) = 1 = \Gamma(2)$. (You do? How?) As x approaches zero, use the relation $\Gamma(x) = \Gamma(x+1)/x$ and because the numerator for small x is approximately 1, you immediately have that

$$\Gamma(x) \sim 1/x \qquad \text{for small } x \tag{1.15}$$

The integral definition, Eq. (1.12), for the Gamma function is defined only for the case that $x > 0$. [The behavior of the integrand near $t = 0$ is approximately t^{x-1}. Integrate *this* from zero to something and see how it depends on x.] Even though the original definition of the Gamma function fails for negative x, you can extend the definition by using Eq. (1.14) to define Γ for negative arguments. What is $\Gamma(-\frac{1}{2})$ for example? Put $x = -\frac{1}{2}$ in Eq. (1.14).

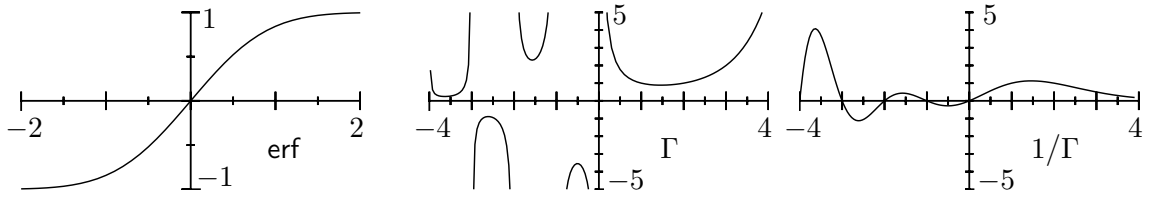
$$-\frac{1}{2}\Gamma(-1/2) = \Gamma(-(1/2)+1) = \Gamma(1/2) = \sqrt{\pi}, \qquad \text{so} \qquad \Gamma(-1/2) = -2\sqrt{\pi} \tag{1.16}$$

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* See for example rpncalculator (v1.96 the latest). It is the best desktop calculator that I've found (Mac and Windows). This main site seems (2008) to have disappeared, but I did find other sources by searching the web for the pair "rpncalculator" and baker. The latter is the author's name. I found mac.rbytes.net/cat/mac/scientific/rpn-calculator-x/

The same procedure works for other negative x, though it can take several integer steps to get to a positive value of x for which you can use the integral definition Eq. (1.12).

The reason for introducing these two functions now is not that they are so much more important than a hundred other functions that I could use, though they are among the more common ones. The point is that the world doesn't end with polynomials, sines, cosines, and exponentials. There are an infinite number of other functions out there waiting for you and some of them are useful. These functions can't be expressed in terms of the elementary functions that you've grown to know and love. They're different and have their distinctive behaviors.



There are zeta functions and Fresnel integrals and Legendre functions and Exponential integrals and Mathieu functions and Confluent Hypergeometric functions and . . . you get the idea. When one of these shows up, you learn to look up its properties and to use them. If you're interested you may even try to understand how some of these properties are derived, but probably not the first time that you confront them. That's why there are tables, and the "Handbook of Mathematical Functions" by Abramowitz and Stegun is a premier example of such a tabulation, and it's reprinted by Dover Publications. There's also a copy on the internet* www.math.sfu.ca/~cbm/aands/ as a set of scanned page images.

Why erf?

What can you do with this function? The most likely application is probably to probability. If you flip a coin 1000 times, you expect it to come up heads *about* 500 times. But just how close to 500 will it be? If you flip it twice, you wouldn't be surprised to see two heads or two tails, in fact the equally likely possibilities are

TT HT TH HH

This says that in 1 out of $2^2 = 4$ such experiments you expect to see two heads and in 1 out of 4 you expect two tails. For just 2 out of 4 times you do the double flip do you expect exactly one head. *All this is an average. You have to try the experiment many times to see your expectation verified, and then only by averaging many experiments.*

It's easier to visualize the counting if you flip N coins at once and see how they come up. The number of coins that come up heads won't always be $N/2$, but it should be close. If you repeat the process, flipping N coins again and again, you get a distribution of numbers of heads that will vary around $N/2$ in a characteristic pattern. The result is that the fraction of the time it will come up with k heads and $N - k$ tails is, to a good approximation

$$\sqrt{\frac{2}{\pi N}}e^{-2\delta^2/N}, \qquad \text{where} \qquad \delta = k - \frac{N}{2} \tag{1.17}$$

The derivation of this can wait until section 2.6, Eq. (2.26). It is an accurate result if the number of coins that you flip in each trial is large, but try it anyway for the preceding example where $N = 2$. This formula says that the fraction of times predicted for k heads is

$$k = 0: \ \sqrt{1/\pi}\, e^{-1} = 0.208 \qquad k = 1 = N/2: \ 0.564 \qquad k = 2: \ 0.208$$

* now superceded by the online reference work dlmf.nist.gov/

The exact answers are $1/4$, $2/4$, $1/4$, but as two is not all that big a number, the fairly large error shouldn't be distressing.

If you flip three coins, the equally likely possibilities are

TTT TTH THT HTT THH HTH HHT HHH

There are 8 possibilities here, 2^3, so you expect (on average) one run out of 8 to give you 3 heads. Probability 1/8.

To see how accurate this claim is for modest values, take $N = 10$. The possible outcomes are anywhere from zero heads to ten. The exact fraction of the time that you get k heads as compared to this approximation is

$k =$	0	1	2	3	4	5
exact:	.000977	.00977	.0439	.117	.205	.246
approximate:	.0017	.0103	.0417	.113	.206	.252

For the more interesting case of big N, the exponent, $e^{-2\delta^2/N}$, varies slowly and smoothly as δ changes in integer steps away from zero. This is a key point; it allows you to approximate a sum by an integral. If $N = 1000$ and $\delta = 10$, the exponent is 0.819. It has dropped only gradually from one. For the same $N = 1000$, the fraction of the time to get exactly 500 heads is 0.025225, and this approximation is $\sqrt{2/1000\pi}$ =0.025231.

Flip N coins, then do it again and again. In what fraction of the trials will the result be between $N/2 - \Delta$ and $N/2 + \Delta$ heads? This is the sum of the fractions corresponding to $\delta = 0$, $\delta = \pm 1$, ..., $\delta = \pm\Delta$. Because the approximate function is smooth, I can replace this sum with an integral. This substitution becomes more accurate the larger N is.

$$\int_{-\Delta}^{\Delta} d\delta \sqrt{\frac{2}{\pi N}} e^{-2\delta^2/N}$$

Make the substitution $2\delta^2/N = x^2$ and you have

$$\sqrt{\frac{2}{\pi N}} \int_{-\Delta\sqrt{2/N}}^{\Delta\sqrt{2/N}} \sqrt{\frac{N}{2}} dx\, e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\Delta\sqrt{2/N}}^{\Delta\sqrt{2/N}} dx\, e^{-x^2} = \mathrm{erf}\left(\Delta\sqrt{2/N}\right) \tag{1.18}$$

The error function of one is 0.84, so if $\Delta = \sqrt{N/2}$ then in 84% of the trials heads will come up between $N/2 - \sqrt{N/2}$ and $N/2 + \sqrt{N/2}$ times. For $N = 1000$, this is between 478 and 522 heads.

1.5 Differentiating

When you differentiate a function in which the independent variable shows up in several places, how do you carry out the derivative? For example, what is the derivative with respect to x of x^x? The answer is that you treat each instance of x one at a time, ignoring the others; differentiate with respect to *that* x and add the results. For a proof, use the definition of a derivative and differentiate the function $f(x, x)$. Start with the finite difference quotient:

$$\begin{aligned} &\frac{f(x+\Delta x, x+\Delta x) - f(x,x)}{\Delta x} \\ &\quad = \frac{f(x+\Delta x, x+\Delta x) - f(x, x+\Delta x) + f(x, x+\Delta x) - f(x,x)}{\Delta x} \\ &\quad = \frac{f(x+\Delta x, x+\Delta x) - f(x, x+\Delta x)}{\Delta x} + \frac{f(x, x+\Delta x) - f(x,x)}{\Delta x} \end{aligned} \tag{1.19}$$

The first quotient in the last equation is, in the limit that $\Delta x \to 0$, the derivative of f with respect to its first argument. The second quotient becomes the derivative with respect to the second argument.

The prescription is clear, but to remember it you may prefer a mathematical formula. A notation more common in mathematics than in physics is just what's needed:

$$\frac{d}{dt}f(t,\,t) = D_1 f(t,\,t) + D_2 f(t,\,t) \tag{1.20}$$

where D_1 means "differentiate with respect to the first argument." The standard "product rule" for differentiation is a special case of this.

For example,

$$\frac{d}{dx}\int_0^x dt\, e^{-xt^2} = e^{-x^3} - \int_0^x dt\, t^2 e^{-xt^2} \tag{1.21}$$

The resulting integral in this example is related to an error function, see problem 1.13, so it's not as bad as it looks.

Another example,

$$\begin{aligned}\frac{d}{dx}x^x &= x\,x^{x-1} + \frac{d}{dx}k^x \qquad \text{at } k = x \\ &= x\,x^{x-1} + \frac{d}{dx}e^{x\ln k} = x\,x^{x-1} + \ln k\, e^{x\ln k} \\ &= x^x + x^x\,\ln x\end{aligned}$$

1.6 Integrals

What is an integral? You've been using them for some time. I've been using the concept in this introductory chapter as if it's something that everyone knows. But what *is* it?

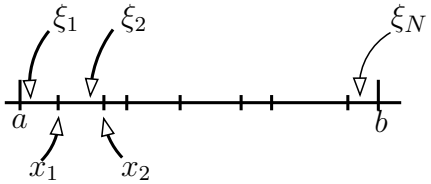
If your answer is something like "the function whose derivative is the given function" or "the area under a curve" then No. Both of these answers express an aspect of the subject but neither is a complete answer. The first actually refers to *the fundamental theorem of calculus,* and I'll describe that shortly. The second is a good picture that applies to some special cases, but it won't tell you how to compute it and it won't allow you to generalize the idea to the many other subjects in which it is needed. There are several different definitions of the integral, and every one of them requires more than a few lines to explain. I'll use the most common definition, the *Riemann Integral*.

An integral is a sum, obeying all the usual rules of addition and multiplication, such as $1+2+3+4 = (1+2)+(3+4)$ or $5\cdot(6+7) = (5\cdot 6)+(5\cdot 7)$. *When you've read this section, come back and translate these bits of arithmetic into statements about integrals.*

A standard way to picture the definition is to try to find the area under a curve. You can get successively better and better approximations to the answer by dividing the area into smaller and smaller rectangles — ideally, taking the limit as the number of rectangles goes to infinity.

To codify this idea takes a sequence of steps:

1. Pick an integer $N > 0$. This is the number of subintervals into which the whole interval between a and b is to be divided.



2. Pick $N-1$ points between a and b. Call them x_1, x_2, etc.

$$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$$

and for convenience label the endpoints as x_0 and x_N. For the sketch , $N = 8$.

3. Let $\Delta x_k = x_k - x_{k-1}$. That is,

$$\Delta x_1 = x_1 - x_0, \qquad \Delta x_2 = x_2 - x_1, \cdots$$

4. In each of the N subintervals, pick one point at which the function will be evaluated. I'll label these points by the Greek letter ξ. (That's the Greek version of "x.")

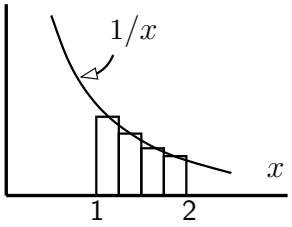
$$x_{k-1} \le \xi_k \le x_k$$
$$x_0 \le \xi_1 \le x_1, \qquad x_1 \le \xi_2 \le x_2, \cdots$$

5. Form the sum that is an approximation to the final answer.

$$f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + f(\xi_3)\Delta x_3 + \cdots$$

6. Finally, take the limit as all the $\Delta x_k \to 0$ and necessarily then, as $N \to \infty$. These six steps form the definition

$$\boxed{\lim_{\Delta x_k \to 0} \sum_{k=1}^{N} f(\xi_k)\Delta x_k = \int_a^b f(x)\,dx} \tag{1.22}$$



To demonstrate this numerically, pick a function and do the first five steps explicitly. Pick $f(x) = 1/x$ and integrate it from 1 to 2. The exact answer is the natural log of 2: $\ln 2 = 0.69315\ldots$
(1) Take $N = 4$ for the number of intervals
(2) Choose to divide the distance from 1 to 2 evenly, at $x_1 = 1.25$, $x_2 = 1.5$, $x_3 = 1.75$

$$a = x_0 = 1. < 1.25 < 1.5 < 1.75 < 2. = x_4 = b$$

(3) All the Δx's are equal to 0.25.
(4) Choose the midpoint of each subinterval. This is the best choice when you use a finite number of divisions without taking the limit.

$$\xi_1 = 1.125 \qquad \xi_2 = 1.375 \qquad \xi_3 = 1.625 \qquad \xi_4 = 1.875$$

(5) The sum approximating the integral is then

$$f(\xi_1)\Delta x_1 \;+\; f(\xi_2)\Delta x_2 \;+\; f(\xi_3)\Delta x_3 \;+\; f(\xi_4)\Delta x_4 \;=$$
$$\frac{1}{1.125} \times .25 + \frac{1}{1.375} \times .25 + \frac{1}{1.625} \times .25 + \frac{1}{1.875} \times .25 = .69122$$

For such a small number of divisions, this is a very good approximation — about 0.3% error. (What do you get if you take $N = 1$ or $N = 2$ or $N = 10$ divisions?)

Fundamental Thm. of Calculus

If the function that you're integrating is complicated or if the function is itself not known to perfect accuracy then a numerical approximation just like this one for $\int_1^2 dx/x$ is often the best way to go. How can a function not be known completely? If it is experimental data. When you have to resort to this arithmetic way to do integrals, are there more efficient ways to do it than simply using the definition of the integral? Yes. That's part of the subject of numerical analysis, and there's a short introduction to the subject in chapter 11, section 11.4.

The fundamental theorem of calculus unites the subjects of differentiation and integration. The integral is defined as the limit of a sum, and the derivative is defined as the limit of a quotient of two differences. The relation between them is

IF f has an integral from a to b, that is, if $\int_a^b f(x)\,dx$ exists,
AND IF f has an anti-derivative, that is, there is a function F such that $dF/dx = f$,
THEN

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{1.23}$$

Are there cases where one of these exists without the other? Yes, though I'll admit that you are not likely to come across such functions without hunting through some advanced math books. Check out www.wikipedia.org for Volterra's function to see what it involves.

Notice an important result that follows from Eq. (1.23). Differentiate both sides with respect to b

$$\frac{d}{db}\int_a^b f(x)\,dx = \frac{d}{db}F(b) = f(b) \tag{1.24}$$

and with respect to a

$$\frac{d}{da}\int_a^b f(x)\,dx = -\frac{d}{da}F(a) = -f(a) \tag{1.25}$$

Differentiating an integral with respect to one or the other of its limits results in plus or minus the integrand. Combine this with the chain rule and you can do such calculations as

$$\frac{d}{dx}\int_{x^2}^{\sin x} e^{xt^2}\,dt = e^{x\sin^2 x}\cos x - e^{x^5}2x + \int_{x^2}^{\sin x} t^2 e^{xt^2}\,dt \tag{1.26}$$

All this requires is that you differentiate every x that is present and add the results, just as

$$\frac{d}{dx}x^2 = \frac{d}{dx}x\cdot x = \frac{dx}{dx}x + x\frac{dx}{dx} = 1\cdot x + x\cdot 1 = 2x$$

You may well ask why anyone would want to do such a thing as Eq. (1.26), but there are more reasonable examples that show up in real situations. I've already used this result in Eq. (1.21).

Riemann-Stieltjes Integrals

Are there other useful definitions of the word integral? Yes, there are many, named after various people who developed them, with Lebesgue being the most famous. His definition* is most useful in much more advanced mathematical contexts, and I won't go into it here, except to say that *very* roughly where Riemann divided the x-axis into intervals Δx_i, Lebesgue divided the y-axis into intervals Δy_i. Doesn't sound like much of a change does it? It is. There is another definition that is worth knowing about, not because it helps you to do integrals, but because it unites a couple of different types of computation into one. This is the *Riemann-Stieltjes* integral. You won't need it for any of the later

* One of the more notable PhD theses in history

work in this book, but it is a fairly simple extension of the Riemann integral and I'm introducing it mostly for its cultural value — to show you that there *are* other ways to define an integral. If you take the time to understand it, you will be able to look back at some subjects that you already know and to realize that they can be manipulated in a more compact form (e.g. center of mass).

When you try to evaluate the moment of inertia you are doing the integral

$$\int r^2\, dm$$

When you evaluate the position of the center of mass even in one dimension the integral is

$$\frac{1}{M}\int x\, dm$$

and even though you may not yet have encountered this, the electric dipole moment is

$$\int \vec{r}\, dq$$

How do you integrate x with respect to m? What exactly are you doing? A possible answer is that you can express this integral in terms of the linear density function and then $dm = \lambda(x)dx$. But if the masses are a mixture of continuous densities and point masses, this starts to become awkward. Is there a better way?

Yes

On the interval $a \le x \le b$ assume there are *two* functions, f and α. Don't assume that either of them must be continuous, though they can't be too badly behaved or nothing will converge. This starts the same way the Riemann integral does: partition the interval into a finite number (N) of sub-intervals at the points

$$a = x_0 < x_1 < x_2 < \ldots < x_N = b \tag{1.27}$$

Form the sum

$$\sum_{k=1}^{N} f(x'_k)\Delta\alpha_k, \qquad \text{where} \qquad x_{k-1} \le x'_k \le x_k \qquad \text{and} \qquad \Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1}) \tag{1.28}$$

To improve the sum, keep adding more and more points to the partition so that in the limit all the intervals $x_k - x_{k-1} \to 0$. This limit is called the Riemann-Stieltjes integral,

$$\int f\, d\alpha \tag{1.29}$$

What's the big deal? Doesn't $d\alpha = \alpha' dx$? Use that and you have just the ordinary integral

$$\int f(x)\alpha'(x)\, dx?$$

Sometimes you can, but what if α isn't differentiable? Suppose that it has a step or several steps? The derivative isn't defined, but this Riemann-Stieltjes integral still makes perfectly good sense.

An example. A very thin rod of length L is placed on the x-axis with one end at the origin. It has a uniform linear mass density λ *and* an added point mass m_0 at $x = 3L/4$. (a piece of chewing gum?) Let $m(x)$ be the function defined as

$$\begin{aligned} m(x) &= \big(\text{the amount of mass at coordinates} \le x\big) \\ &= \begin{cases} \lambda x & (0 \le x < 3L/4) \\ \lambda x + m_0 & (3L/4 \le x \le L) \end{cases} \end{aligned}$$

This is of course discontinuous.



The coordinate of the center of mass is $\int x\,dm / \int dm$. The total mass in the denominator is $m_0 + \lambda L$, and I'll go through the details to evaluate the numerator, attempting to solidify the ideas that form this integral. Suppose you divide the length L into 10 equal pieces, then

$$x_k = kL/10, \quad (k = 0, 1, \ldots, 10) \qquad \text{and} \qquad \Delta m_k = \begin{cases} \lambda L/10 & (k \ne 8) \\ \lambda L/10 + m_0 & (k = 8) \end{cases}$$

$\Delta m_8 = m(x_8) - m(x_7) = (\lambda x_8 + m_0) - \lambda x_7 = \lambda L/10 + m_0$.

Choose the positions x'_k anywhere in the interval; for no particular reason I'll take the right-hand endpoint, $x'_k = kL/10$. The approximation to the integral is now

$$\begin{aligned} \sum_{k=1}^{10} x'_k \Delta m_k &= \sum_{k=1}^{7} x'_k \lambda L/10 \;+\; x'_8(\lambda L/10 + m_0) \;+\; \sum_{k=9}^{10} x'_k \lambda L/10 \\ &= \sum_{k=1}^{10} x'_k \lambda L/10 \;+\; x'_8 m_0 \end{aligned}$$

As you add division points (more intervals) to the whole length this sum obviously separates into two parts. One is the ordinary integral and the other is the discrete term from the point mass.

$$\int_0^L x\lambda\,dx + m_0 3L/4 = \lambda L^2/2 + m_0 3L/4$$

The center of mass is then at

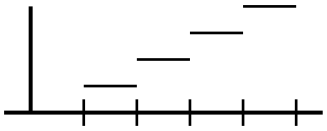
$$x_{\text{cm}} = \frac{\lambda L^2/2 + m_0 3L/4}{m_0 + \lambda L}$$

If $m_0 \ll \lambda L$, this is approximately $L/2$. In the reverse case is is approximately $3L/4$. Both are just what you should expect.

The discontinuity in $m(x)$ simply gives you a discrete added term in the overall result.

Did you need the Stieltjes integral to do this? Probably not. You would likely have simply added the two terms from the two parts of the mass and gotten the same result as with this more complicated method. The point of this is not that it provides an easier way to do computations. It doesn't. It is

however a unifying notation and language that lets you avoid writing down a lot of special cases. (Is it discrete? Is it continuous?) You can even write sums as integrals: Let α be a set of steps:



$$\alpha(x) = \begin{cases} 0 & x < 1 \\ 1 & 1 \le x < 2 \\ 2 & 2 \le x < 3 \\ \text{etc.} \end{cases} \qquad = [x] \quad \text{for } x \ge 0$$

Where that last bracketed symbol means "greatest integer less than or equal to x." It's a notation more common in mathematics than in physics. Now in this notation the sum can be written as a Stieltjes integral.

$$\int f\, d\alpha = \int_{x=0}^{\infty} f\, d[x] = \sum_{k=1}^{\infty} f(k) \tag{1.30}$$

At every integer, where $[x]$ makes a jump by one, there is a contribution to the Riemann-Stieltjes sum, Eq. (1.28). That makes this integral just another way to write the sum over integers. This won't help you to sum the series, but it is another way to look at the subject.

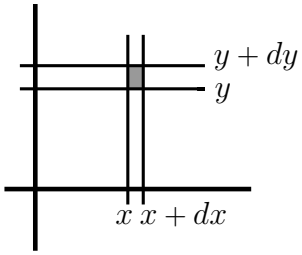
The method of integration by parts works perfectly well here, though as with all the rest of this material I'll leave the proof to advanced calculus texts. If $\int f\, d\alpha$ exists then so does $\int \alpha\, df$ and

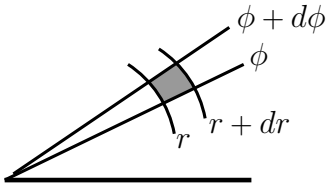
$$\int f\, d\alpha = f\alpha - \int \alpha\, df \tag{1.31}$$

This relates one Stieltjes integral to another one, and because you can express summation as an integral now, you can even do summation by parts on the equation (1.30). That's something that you are not likely to think of if you restrict yourself to the more elementary notation, and it's even occasionally useful.

1.7 Polar Coordinates

When you compute an integral in the plane, you need the element of area appropriate to the coordinate system that you're using. In the most common case, that of rectangular coordinates, you find the element of area by drawing the two lines at constant coordinates x and $x + dx$. Then you draw the two lines at constant coordinates y and $y + dy$. The little rectangle that they circumscribe has an area $dA = dx\, dy$.





In polar coordinates you do exactly the same thing! The coordinates are r and ϕ, and the line at constant radius r and at constant $r + dr$ define two neighboring circles. The lines at constant angle ϕ and at constant angle $\phi + d\phi$ form two closely spaced rays from the origin. These four lines circumscribe a tiny area that is, for small enough dr and $d\phi$, a rectangle. You then know its area is the product of its two sides*: $dA = (dr)(r\, d\phi)$. This is the basic element of area for polar coordinates.

* If you're tempted to say that the area is $dA = dr\, d\phi$, *look at the dimensions.* This expression is a length, not an area.

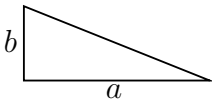
The area of a circle is the sum of all the pieces of area within it

$$\int dA = \int_0^R r\,dr \int_0^{2\pi} d\phi$$

I find it more useful to write double integrals in this way, so that the limits of integration are next to the differential. The other notation can put the differential a long distance from where you show the limits of integration. I get less confused my way. In either case, and to no one's surprise, you get

$$\int_0^R r\,dr \int_0^{2\pi} d\phi = \int_0^R r\,dr\,2\pi = 2\pi R^2/2 = \pi R^2$$

For the preceding example you can do the double integral in either order with no special care. If the area over which you're integrating is more complicated you will have to look more closely at the limits of integration. I'll illustrate with an example of this in rectangular coordinates: the area of a triangle. Take the triangle to have vertices $(0,0)$, $(a,0)$, and $(0,b)$. The area is



$$\int dA = \int_0^a dx \int_0^{b(a-x)/a} dy \qquad \text{or} \qquad \int_0^b dy \int_0^{a(b-y)/b} dx \tag{1.32}$$

They should both yield $ab/2$. See problem 1.25.

1.8 Sketching Graphs

How do you sketch the graph of a function? This is one of the most important tools you can use to understand the behavior of functions, and unless you practice it you will find yourself at a loss in anticipating the outcome of many calculations. There are a handful of rules that you can follow to do this and you will find that it's not as painful as you may think.

You are confronted with a function and have to sketch its graph.

1. What is the domain? That is, what is the set of values of the independent variable that you need to be concerned with? Is it $-\infty$ to $+\infty$ or is it $0 < x < L$ or is it $-\pi < \phi < \pi$ or what?

2. Plot any obvious points. If you can immediately see the value of the function at one or more points, do them right away.

3. Is the function even or odd? If the behavior of the function is the same on the left as it is on the right (or perhaps inverted on the left) then you have half as much work to do. Concentrate on one side and you can then make a mirror image on the left if it is even or an upside-down mirror image if it's odd.

4. Is the function singular anywhere? Does it go to infinity at some point where the denominator vanishes? Note these points on the axis for future examination.

5. What is the behavior of the function *near* any of the obvious points that you plotted? Does it behave like x? Like x^2? If you concluded that it is even, then the slope is either zero or there's a kink in the curve, such as with the absolute value function, $|x|$.

6. At one of the singular points that you found, how does it behave as you approach the point from the right? From the left? Does the function go toward $+\infty$ or toward $-\infty$ in each case?

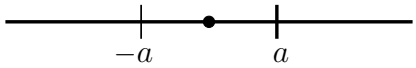
7. How does the function behave as you approach the ends of the domain? If the domain extends from $-\infty$ to $+\infty$, how does the function behave as you approach these regions?

8. Is the function the sum or difference of two other much simpler functions? If so, you may find it easier to sketch the two functions and then graphically add or subtract them. Similarly if it is a product.

9. Is the function related to another by translation? The function $f(x) = (x-2)^2$ is related to x^2 by translation of 2 units. Note that it is translated to the *right* from x^2. You can see why because $(x-2)^2$ vanishes at $x = +2$.

10. After all this, you will have a good idea of the shape of the function, so you can interpolate the behavior between the points that you've found.

Example: sketch $f(x) = x/(a^2 - x^2)$.

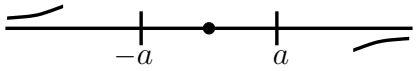


1. The domain for independent variable wasn't given, so take it to be $-\infty < x < \infty$

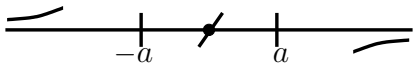
2. The point $x = 0$ obviously gives the value $f(0) = 0$.

4. The denominator becomes zero at the two points $x = \pm a$.

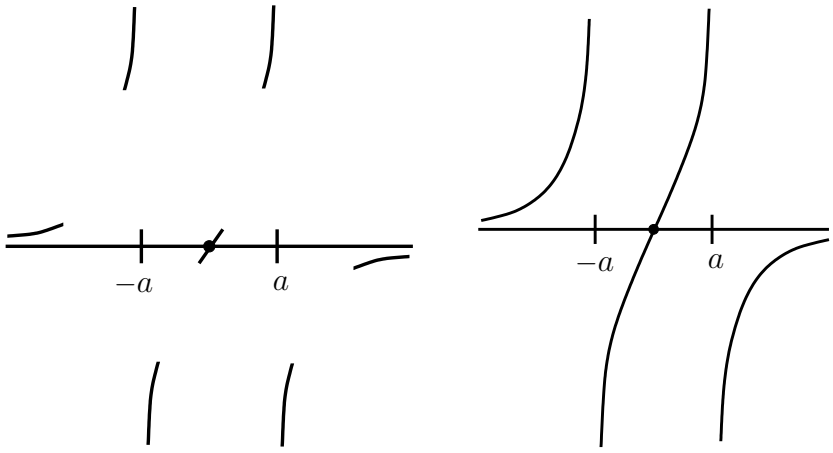
3. If you replace x by $-x$, the denominator is unchanged, and the numerator changes sign. The function is odd about zero.



7. When x becomes very large ($|x| \gg a$), the denominator is mostly $-x^2$, so $f(x)$ behaves like $x/(-x^2) = -1/x$ for large x. It approaches zero for large x. Moreover, when x is positive, it approaches zero through negative values and when x is negative, it goes to zero through positive values.



5. Near the point $x = 0$, the x^2 in the denominator is much smaller than the constant a^2 ($x^2 \ll a^2$). That means that near this point, the function f behaves like x/a^2



6. Go back to the places that it blows up, and ask what happens near there. If x is a little greater than a, the x^2 in the denominator is a little larger than the a^2 in the denominator. This means that the denominator is negative. When x is a little less than a, the reverse is true. Near $x = a$, The numerator is close to a. Combine these, and you see that the function approaches $-\infty$ as $x \to a$ from the right. It approaches $+\infty$ on the left side of a. I've already noted that the function is odd, so don't repeat the analysis near $x = -a$, just turn this behavior upside down.

With all of these pieces of the graph, you can now interpolate to see the whole picture.

OR, if you're clever with partial fractions, you might realize that you can rearrange f as

$$\frac{x}{a^2 - x^2} = \frac{-1/2}{x - a} + \frac{-1/2}{x + a},$$

and then follow the ideas of techniques 8 and 9 to sketch the graph. It's not obvious that this is any easier; it's just different.

Exercises

1 Express e^x in terms of hyperbolic functions.

2 If $\sinh x = 4/3$, what is $\cosh x$? What is $\tanh x$?

3 If $\tanh x = 5/13$, what is $\sinh x$? What is $\cosh x$?

4 Let n and m be positive integers. Let $a = n^2 - m^2$, $b = 2nm$, $c = n^2 + m^2$. Show that a-b-c form the integer sides of a right triangle. What are the first three independent "Pythagorean triples?" By that I mean ones that aren't just a multiple of one of the others.

5 Evaluate the integral $\int_0^a dx\, x^2 \cos x$. Use parametric differentiation starting with $\cos \alpha x$.

6 Evaluate $\int_0^a dx\, x \sinh x$ by parametric differentiation.

7 Differentiate $xe^x \sin x \cosh x$ with respect to x.

8 Differentiate $\int_0^{x^2} dt\, \sin(xt)$ with respect to x.

9 Differentiate $\int_{-x}^{+x} dt\, e^{-xt^4}$ with respect to x.

10 Differentiate $\int_{-x}^{+x} dt\, \sin(xt^3)$ with respect to x.

11 Differentiate $\int_0^{\sqrt[3]{\sin(kx)}} dt\, e^{-\alpha t^3} J_0(\beta t)$ with respect to x. J_0 is a Bessel function.

12 Sketch the function $y = v_0 t - gt^2/2$. (First step: set all constants to one. $v_0 = g = 2 = 1$. Except exponents)

13 Sketch the function $U = -mgy + ky^2/2$. (Again: set the constant factors to one.)

14 Sketch $U = mg\ell(1 - \cos\theta)$.

15 Sketch $V = -V_0 e^{-x^2/a^2}$.

16 Sketch $x = x_0 e^{-\alpha t} \sin \omega t$.

17 Is it all right in Eq. (1.22) to replace "$\Delta x_k \to 0$" with "$N \to \infty$?" [No.]

18 Draw a graph of the curve parametrized as $x = \cos\theta$, $y = \sin\theta$.
Draw a graph of the curve parametrized as $x = \cosh\theta$, $y = \sinh\theta$.

19 What is the integral $\int_a^b dx\, e^{-x^2}$?

20 Given that $\int_{-\infty}^{\infty} dx/(1+x^2) = \pi$, *i.e.* you don't have to derive this, what then is $\int_{-\infty}^{\infty} dx/(\alpha+x^2)$? Now differentiate the result and find the two integrals $\int_{-\infty}^{\infty} dx/(1+x^2)^2$ and $\int_{-\infty}^{\infty} dx/(1+x^2)^3$.

21 Derive the product rule as a special case of Eq. (1.20).

22 The third paragraph of section 1.6 has two simple equations in arithmetic. What common identities about the integral do these correspond to?

23 Plot a graph of $y = e^x$ with y and x in meters (x horizontal and y vertical). Start at the origin and walk along the x-axis at one meter per second. When you are at the 20-meter point, where is the y coordinate and how fast is it rising? Not just numbers: compare both to real things.

Problems

1.1 What is the tangent of an angle in terms of its sine? Draw a triangle and do this in one line.

1.2 Derive the identities for $\cosh^2\theta - \sinh^2\theta$ and for $1 - \tanh^2\theta$, Equation (1.3).

1.3 Derive the expressions in Eq. (1.4) for $\cosh^{-1} y$, $\tanh^{-1} y$, and $\coth^{-1} y$. Pay particular attention to the domains and explain why these are valid for the set of y that you claim. What is $\sinh^{-1}(y) + \sinh^{-1}(-y)$?

1.4 The inverse function has a graph that is the mirror image of the original function in the $45°$ line $y = x$. Draw the graphs of all six of the hyperbolic functions and all six of the inverse hyperbolic functions, comparing the graphs you should get to the functions derived in the preceding problem.

1.5 Evaluate the derivatives of $\cosh x$, $\tanh x$, and $\coth x$.

1.6 What are the derivatives, $d\sinh^{-1} y/dy$ and $d\cosh^{-1} y/dy$?

1.7 Find formulas for $\sinh 2y$ and $\cosh 2y$ in terms of hyperbolic functions of y. The first one of these should take just a couple of lines. Maybe the second one too, so if you find yourself filling a page, start over.

1.8 Do a substitution to evaluate the integral (**a**) simply. Now do the same for (**b**)

$$\text{(a)} \quad \int \frac{dt}{\sqrt{a^2 - t^2}} \qquad \text{(b)} \quad \int \frac{dt}{\sqrt{a^2 + t^2}}$$

1.9 Sketch the two integrands in the preceding problem. For the second integral, if the limits are 0 and z with $z \gg a$, then before having done the integral, estimate *approximately* what the value of this integral should be. (Say $z = 10^6 a$ or $z = 10^{60} a$.) Compare your estimate to the exact answer that you just found to see if they match in any way.

1.10 Fill in the steps in the derivation of the Gaussian integrals, Eqs. (1.7), (1.8), and (1.10). In particular, draw graphs of the integrands to show why Eq. (1.7) is so.

1.11 What is the integral $\int_{-\infty}^{\infty} dt\, t^n e^{-t^2}$ if $n = -1$ or $n = -2$? [*Careful!*, no conclusion-jumping allowed.] Did you draw a graph? No? Then that's why you're having trouble with this.

1.12 Sketch a graph of the error function. In particular, what is its behavior for small x and for large x, both positive and negative? Note: "small" doesn't mean zero. First draw a sketch of the integrand e^{-t^2} and from that you can (graphically) estimate $\mathrm{erf}(x)$ for small x. Compare this to the short table in Eq. (1.11).

1.13 Put a parameter α into the defining integral for the error function, Eq. (1.11), so it has $\int_0^x dt\, e^{-\alpha t^2}$. Differentiate this integral with respect to α. Next, change variables in this same integral from t to u: $u^2 = \alpha t^2$, and differentiate *that* integral (which of course has the same value as before) with respect to alpha to show

$$\int_0^x dt\, t^2 e^{-t^2} = \frac{\sqrt{\pi}}{4}\,\mathrm{erf}(x) - \frac{1}{2} x e^{-x^2}$$

As a check, does this agree with the previous result for $x = \infty$, Eq. (1.10)?

1.14 Use parametric differentiation to derive the recursion relation $x\Gamma(x) = \Gamma(x+1)$. Do it once by inserting a parameter in the integral for Γ, $e^{-t} \to e^{-\alpha t}$, and differentiating. Then change variables before differentiating and equate the results.

1.15 What is the Gamma function of $x = -1/2$, $-3/2$, $-5/2$? Explain why the original definition of Γ in terms of the integral won't work here. Demonstrate why Eq. (1.12) converges for all $x > 0$ but does not converge for $x \le 0$. Ans: $\Gamma(-5/2) = -8\sqrt{\pi}/15$

1.16 What is the Gamma function for x near 1? near 0? near -1? -2? -3? Now sketch a graph of the Gamma function from -3 through positive values. Try using the recursion relation of problem 1.14. Ans: Near -3, $\Gamma(x) \approx -1/\big(6(x+3)\big)$

1.17 Show how to express the integral for arbitrary positive x

$$\int_0^\infty dt\, t^x e^{-t^2}$$

in terms of the Gamma function. Is *positive* x the best constraint here or can you do a touch better? Ans: $\frac{1}{2}\Gamma\big((x+1)/2\big)$

1.18 The derivative of the Gamma function at $x = 1$ is $\Gamma'(1) = -0.5772 = -\gamma$. The number γ is called Euler's constant, and like π or e it's another number that simply shows up regularly. What is $\Gamma'(2)$? What is $\Gamma'(3)$? Ans: $\Gamma'(3) = 3 - 2\gamma$

1.19 Show that

$$\Gamma(n + {}^1\!/_2) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!$$

The "double factorial" symbol mean the product of every other integer up to the given one. E.g. $5!! = 15$. The double factorial of an even integer can be expressed in terms of the single factorial. Do so. What about odd integers?

1.20 Evaluate this integral. Just find the right substitution. $\displaystyle\int_0^\infty dt\, e^{-t^a} \qquad (a > 0)$

1.21 A triangle has sides a, b, c, and the angle opposite c is γ. Express the area of the triangle in terms of a, b, and γ. Write the law of cosines for this triangle and then use $\sin^2\gamma + \cos^2\gamma = 1$ to express the area of a triangle solely in terms of the lengths of its three sides. The resulting formula is not especially pretty or even clearly symmetrical in the sides, but if you introduce the semiperimeter, $s = (a+b+c)/2$, you can rearrange the answer into a neat, symmetrical form. Check its validity in a couple of special cases. Ans: $\sqrt{s(s-a)(s-b)(s-c)}$ (Hero's formula)

1.22 An arbitrary linear combination of the sine and cosine, $A\sin\theta + B\cos\theta$, is a phase-shifted cosine: $C\cos(\theta+\delta)$. Solve for C and δ in terms of A and B, deriving an identity in θ.

1.23 Solve the two simultaneous linear equations

$$ax + by = e, \qquad cx + dy = f$$

and do it solely by elementary manipulation ($+$, $-$, $\times$, $\div$), not by any special formulas. Analyze all the *qualitatively different* cases and draw graphs to describe each. In every case, how many if any solutions

are there? Because of its special importance later, look at the case $e = f = 0$ and analyze it as if it's a separate problem. You should be able to discern and to classify the circumstances under which there is one solution, no solution, or many solutions. Ans: Sometimes a unique solution. Sometimes no solution. Sometimes many solutions. Draw two lines in the plane; how many qualitatively different pictures are there?

1.24 Use parametric differentiation to evaluate the integral $\int x^2 \sin x \, dx$. Find a table of integrals if you want to verify your work.

1.25 Derive all the limits on the integrals in Eq. (1.32) and then do the integrals.

1.26 Compute the area of a circle using rectangular coordinates,

1.27 (**a**) Compute the area of a triangle using rectangular coordinates, so $dA = dx\,dy$. Make it a right triangle with vertices at $(0,0)$, $(a,0)$, and (a,b). (**b**) Do it again, but reversing the order of integration. (**c**) Now compute the area of this triangle using *polar* coordinates. Examine this carefully to see which order of integration makes the problem easier.

1.28 Start from the definition of a derivative, $\lim \big(f(x+\Delta x) - f(x)\big)/\Delta x$, and derive the chain rule.

$$f(x) = g\big(h(x)\big) \Longrightarrow \frac{df}{dx} = \frac{dg}{dh}\frac{dh}{dx}$$

Now pick special, fairly simple cases for g and h to test whether your result really works. That is, choose functions so that you can do the differentiation explicitly and compare the results, but also functions with enough structure that they aren't trivial.

1.29 Starting from the definitions, derive how to do the derivative,

$$\frac{d}{dx}\int_0^{f(x)} g(t)\,dt$$

Now pick special, fairly simple cases for f and g to test whether your result really works. That is, choose functions so that you can do the integration and differentiation explicitly, but ones such the result isn't trivial.

1.30 Sketch these graphs, working by hand only, no computers:

$$\frac{x}{a^2+x^2}, \qquad \frac{x^2}{a^2-x^2}, \qquad \frac{x}{a^3+x^3}, \qquad \frac{x-a}{a^2-(x-a)^2}, \qquad \frac{x}{L^2-x^2} + \frac{x}{L}$$

1.31 Sketch by hand only, graphs of

$$\sin x \ (-3\pi < x < +4\pi), \qquad \frac{1}{\sin x} \ (-3\pi < x < +4\pi), \qquad \sin(x - \pi/2) \ (-3\pi < x < +4\pi)$$

1.32 Sketch by hand only, graphs of

$$f(\phi) = 1 + \frac{1}{2}\sin^2\phi \ (0 \le \phi \le 2\pi), \qquad f(\phi) = \begin{cases} \phi & (0 < \phi < \pi) \\ \phi - 2\pi & (\pi < \phi < 2\pi) \end{cases}$$

$$f(x) = \begin{cases} x^2 & (0 \le x < a) \\ (x-2a)^2 & (a \le x \le 2a) \end{cases}, \qquad f(r) = \begin{cases} Kr/R^3 & (0 \le r \le R) \\ K/r^2 & (R < r < \infty) \end{cases}$$

1.33 From the definition of the Riemann integral make a numerical calculation of the integral

$$\int_0^1 dx\,\frac{4}{1+x^2}$$

Use 1 interval, then 2 intervals, then 4 intervals. If you choose to write your own computer program for an arbitrary number of intervals, by all means do so. As with the example in the text, choose the midpoints of the intervals to evaluate the function. To check your answer, do a trig substitution and evaluate the integral exactly. What is the % error from the exact answer in each case? [100×(wrong − right)/right] Ans: π

1.34 Evaluate $\mathrm{erf}(1)$ numerically. Use 4 intervals. Ans: 0.842700792949715 (more or less)

1.35 Evaluate $\int_0^\pi dx\,\sin x/x$ numerically. Ans: 1.85193705198247 or so.

1.36 x and y are related by the equation $x^3-4xy+3y^3=0$. You can easily check that $(x,y)=(1,1)$ satisfies it, now what is dy/dx at that point? Unless you choose to look up and plug in to the cubic formula, I suggest that you differentiate the whole equation with respect to x and solve for dy/dx. Generalize this to finding dy/dx if $f(x,y)=0$. Ans: $1/5$

1.37 When flipping a coin N times, what fraction of the time will the number of heads in the run lie between $\left(N/2-2\sqrt{N/2}\right)$ and $\left(N/2+2\sqrt{N/2}\right)$? What are these numbers for $N=1000$? Ans: 99.5%

1.38 For $N=4$ flips of a coin, count the number of times you get 0, 1, 2, *etc.* heads out of $2^4=16$ cases. Compare these results to the exponential approximation of Eq. (1.17).
Ans: $2\to$ 0.375 and 0.399

1.39 Is the integral of Eq. (1.17) over all δ equal to one?

1.40 If there are 100 molecules of a gas bouncing around in a room, about how long will you have to wait to find that all of them are in the left half of the room? Assume that you make a new observation every microsecond and that the observations are independent of each other. Ans: A million times the age of the universe. [Care to try 10^{23} molecules?]

1.41 If you flip 1000 coins 1000 times, about how many times will you get exactly 500 heads and 500 tails? What if it's 100 coins and 100 trials, getting 50 heads? Ans: 25, 8

1.42 (**a**) Use parametric differentiation to evaluate $\int x\,dx$. Start with $\int e^{\alpha x}dx$. Differentiate and then let $\alpha\to 0$.
(**b**) Now that the problem has blown up in your face, change the integral from an indefinite to a definite integral such as $\int_a^b$ and do it again. There are easier ways to do this integral, but the point is that this method is really designed for definite integrals. It may not work on indefinite ones.

1.43 The Gamma function satisfies the identity

$$\Gamma(x)\Gamma(1-x)=\pi/\sin\pi x$$

What does this tell you about the Gamma function of 1/2? What does it tell you about its behavior near the negative integers? Compare this result to that of problem 1.16.

1.44 Start from the definition of a derivative, manipulate some terms: (**a**) derive the rule for differentiating the function h, where $h(x) = f(x)g(x)$ is the product of two other functions.
(**b**) Integrate the resulting equation with respect to x and derive the formula for integration by parts.

1.45 Show that in polar coordinates the equation $r = 2a\cos\phi$ is a circle. Now compute its area in this coordinate system.

1.46 The cycloid* has the parametric equations $x = a\theta - a\sin\theta$, and $y = a - a\cos\theta$. Compute the area, $\int y\,dx$ between one arc of this curve and the x-axis. Ans: $3\pi a^2$

1.47 An alternate approach to the problem 1.13: Change variables in the integral definition of erf to $t = \alpha u$. Now differentiate with respect to α and of course the derivative must be zero and there's your answer. Do the same thing for problem 1.14 and the Gamma function.

1.48 Recall section 1.5 and compute this second derivative to show

$$\frac{d^2}{dt^2}\int_0^t dt'\,(t-t')F(t') = F(t)$$

1.49 From the definition of a derivative show that

$$\text{If}\quad x = f(\theta)\qquad\text{and}\qquad t = g(\theta)\qquad\text{then}\qquad \frac{dx}{dt} = \frac{df/d\theta}{dg/d\theta}$$

Make up a couple of functions that let you test this explicitly.

1.50 Redo problem 1.6 another way: $x = \sinh^{-1} y \leftrightarrow y = \sinh x$. Differentiate the second of these with respect to y and solve for dx/dy. Ans: $d\sinh^{-1} y/dy = 1/\sqrt{1+y^2}$.

* www-groups.dcs.st-and.ac.uk/~history/Curves/Cycloid.html